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Scale Invariance, Killing Vectors, and the Size of the Fifth Dimension

D. K. Ross[1]

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An analysis is made of the classical five-dimensional sourceless Kaluza-Klein equations with the existence of the usual $\partial/\partial\psi$ Killing vector not assumed, where ψ is the coordinate of the fifth dimension. The physical distance around the fifth dimension D_5, needed for the calculation of the fine structure constant α, is not calculable in the usual theory because the equations have a global scale invariance. In the present case, the Killing vector and the global scale invariance are not present, but it is found rather generally that $D_5 = 0$. This indicates that quantum gravity is a necessary ingredient if α is to be calculated. It also provides an alternate explanation of why the universe appears four-dimensional.

1. INTRODUCTION

Kaluza (1921) and Klein (1926) first wrote down a five-dimensional unification of general relativity and electromagnetism. An important aspect of their work is the assumption that the five-dimensional metric g_{AB} is independent of the coordinate of the fifth dimension ψ. A conceptual problem is why such a $\partial/\partial\psi$ Killing vector should exist. Chodos and Detweiler (1980) have written an interesting paper showing, in the context of a specific Kasner (1921) solution to the five-dimensional vacuum Einstein equations, that three spatial dimensions expand and one contracts to a very small value, explaining why the universe appears to be four-dimensional. They put the electromagnetic vector potential A_μ into $g_{\mu 5}$ as a perturbation and look at the Klein-Gordon equation in five dimensions. As in the earlier work, of Souriau (1963), they find that the fine structure constant α can be written in terms of the distance around the fifth dimension D_5 as $\alpha = (4\pi r_{\text{Planck}}/D_5)^2$, assuming that this fifth dimension is compact. Putting in

[1]Physics Department, Iowa State University Ames, Iowe 50011.

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the experimental value of α gives $D_5 \approx 100 r_{\text{Planck}}$, where $r_{\text{Planck}} = (\hbar G/c^3)^{1/2}$ is the Planck radius. [See Gross and Perry (1983), Pollard (1983), Belinski and Ruffini (1980), Perry (1984), and Sorkin (1983) for a representative sample of other solutions to the Kaluza-Klein equations.]

Chodos and Detweiler (1980) could not calculate D_5 and hence α from first principles because the classical equations with the $\partial/\partial\psi$ Killing vector present are globally scale invariant under a scale transformation involving the ψ coordinate (Gross and Perry, 1983). There is nothing in the theory to set this scale. One possibility explored by Appelquist and Chodos (1983) is that this dilatation invariance is broken once ψ-dependent quantum corrections are taken into account. A calculation of the dilaton mass could fix a value for D_5. They found that they could get quantum shrinkage of an already existing S^1 fifth dimension down to sizes on the order of the Planck length, but they could not calculate D_5.

Another possibility is to abandon the assumption that a $\partial/\partial\psi$ Killing vector exists. Since this assumption partially ruins the unification of gravitation and electromagnetism in a five-dimensional framework, abandoning it might prove fruitful. Thus, in this paper, I consider a five-dimensional space-time with $g_{AB}(\psi)$ and see what can be said about the distance around the fifth dimension D_5 under rather general assumptions. I want to see if the lack of a $\partial/\partial\psi$ Killing vector and hence a lack of scale invariance in the classical equations will make it possible to calculate D_5 and hence α. I find, rather surprisingly, that the ψ dependence in g_{AB} forces D_5 to be *zero* classically. The fifth dimension is unobservable not because it shrinks in time, but rather because the geodesic distance around it vanishes.

I delimit g_{AB} to a form compatible with the observed large-scale universe and write down the five-dimensional Einstein field equations in Section 2. In Section 3 I separate and solve these equations under very general assumptions, and in Section 4 calculate D_5 and summarize the results.

2. FORM OF THE METRIC CONSIDERED AND THE FIELD EQUATIONS

A completely general five-dimensional manifold is much too general for our purposes. We want to limit g_{AB}, where A, B range from 1 through 5, to a form consistent with the form of the large-scale universe. I use a $(-1, 1, 1, 1, 1)$ signature, so that the fifth dimension is spatial. Since the universe is homogeneous and isotropic in its three spatial dimensions, I assume that our five-dimensional space-time has a maximally symmetric three-dimensional subspace. From Weinberg (1972) this simplifies the line element to the form

$$ds^2 = g_{ab}(t, \psi) + f(t, \psi)[dr^2(1-kr^2)^{-1} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2] \tag{1}$$

where $k = 0, \pm 1$ and a and b range over the remaining two dimensions. In addition, I assume that we have a global time coordinate that can be used as the time coordinate of a Gaussian coordinate system. The time coordinate t will thus be taken to be the length of a timelike geodesic that goes through each point and is orthogonal to a four-dimensional spacelike hypersurface. The four spatial coordinates are being treated the same here. The line element then assumes the form

$$ds^2 = -dt^2 + N(t, \psi)\, d\psi^2 + R^2(t,\psi)[dr^2(1-kr^2)^{-1} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2] \tag{2}$$

where r is a dimensionless distance marker. I assume that the topology of the fifth dimension is that of a circle (Bergmann, 1942) and that the circle is small, so that distances in our usual four dimensions do not depend upon the fifth dimension in an unacceptable way. Note that g_{AB} is diagonal and depends explicitly on the coordinate of the fifth dimension ψ. The components of g^{AB} are then

$$g^{tt} = -1, \qquad g^{kj} = R^{-2}\tilde{g}^{kj}, \qquad g^{55} = N^{-1} \tag{3}$$

where $\tilde{g}_{kj}$ refers to the three-dimensional spatial metric inside the brackets in (2). Lowercase Latin letters run over the three coordinates of ordinary space and t always refers to time.

Note that the A_μ of electromagnetism can be put in as a perturbation on $g_{\mu 5}$ later if we wish to relate the fine structure constant to D_5 as Chodos and Detweiler (1980) do (Greek letters refer to four-dimensional space-time). We are assuming in (2) that electromagnetic fields are not playing a large role in the structure of the universe as a whole.

In order to write down the Einstein field equations, we need an energy-momentum tensor. In the usual Kaluza-Klein theory with A_μ present and a $\partial/\partial\psi$ Killing vector assumed, the five-dimensional vacuum Einstein equations, with energy-momentum tensor $T_{AB} = 0$, yield the correct Einstein equations correctly sourced with the electromagnetic $T_{\mu\nu}$ in four dimensions. Thus $T_{AB} = 0$ is a reasonable and logical choice. We could also consider an explicit T_{AB} which is a generalization to five dimensions of the four-dimensional energy mementum tensor for a comoving perfect fluid characterized by a pressure $P(t)$ and density $\rho(t)$. This would modify the time dependence of the field equations below but would not materially modify the ψ-dependent equations or the calculation of D_5. For simplicity and for ease of comparison with the work of Chodos and Detweiler (1980) we consider $T_{AB} = 0$.

From (2) and (3) we can work out the nonzero Christoffel symbols as

$$\begin{aligned}
&\Gamma^t_{ij} = R\dot{R}\tilde{g}_{ij}, \qquad \Gamma^i_{tj} = (\dot{R}/R)\delta^i_j \\
&\Gamma^i_{kj} = \tilde{\Gamma}^i_{kj}, \qquad \Gamma^5_{ij} = -(RR'/N)\tilde{g}_{ij} \\
&\Gamma^i_{5j} = (R'/R)\delta^i_j, \Gamma^5_{55} = \tfrac{1}{2}N'/N \\
&\Gamma^5_{5t} = \tfrac{1}{2}\dot{N}/N, \qquad \Gamma^t_{55} = \dot{N}/2
\end{aligned} \tag{4}$$

where a dot is a time derivative, a prime is a ψ derivative, and a quantity with a tilde above it refers to the spatial three-dimensional metric in $[\cdots]$ brackets in (2).

Using (4), we find that the Einstein equations in five dimensions,

$$R_{AB} = 0 \tag{5}$$

become

$$\frac{1}{2}\frac{\ddot{N}}{N} - \frac{1}{4}\left(\frac{\dot{N}}{N}\right)^2 + \frac{3\ddot{R}}{R} = 0 \tag{6}$$

$$-2k + \frac{2R'^2}{N} - 2\dot{R}^2 - R\ddot{R} + \frac{RR''}{N} - \frac{1}{2}\frac{R\dot{R}\dot{N}}{N} - \frac{1}{2}\frac{RR'N'}{N^2} = 0 \tag{7}$$

$$\frac{3R''}{R} - \frac{\ddot{N}}{2} + \frac{\dot{N}^2}{N}\frac{1}{4} - \frac{3}{2}\frac{R'N'}{RN} - \frac{3}{2}\frac{\dot{R}\dot{N}}{R} = 0 \tag{8}$$

$$\frac{3\dot{R}'}{R} - \frac{3}{2}\frac{\dot{N}}{N}\frac{R'}{R} = 0 \tag{9}$$

for the tt, ij, 55, and $5t$ sectors, respectively. The other sectors give equations which are identically satisfied. These equations are solved in the next section.

3. SEPARATION AND SOLUTION OF THE FIELD EQUATIONS

Let us write

$$N(\psi, t) = F(\psi)T(t), \qquad R(\psi, t) = H(\psi)S(t) \tag{10}$$

and see if we can separate the field equations into equations that depend only on t and equations that depend only on ψ. Substituting (10) into (6)-(9) gives

$$\frac{1}{2}\frac{\ddot{T}}{T} - \frac{1}{4}\left(\frac{\dot{T}}{T}\right)^2 + 3\frac{\ddot{S}}{S} = 0 \tag{11}$$

$$\frac{-2k}{H^2}\frac{T}{S^2} + \frac{2}{F}\left(\frac{H'}{H}\right)^2 + \frac{H''}{HF} - \frac{1}{2}\frac{H'}{H}\frac{F'}{F^2} = T\left[\frac{\ddot{S}}{S} + 2\left(\frac{\dot{S}}{S}\right)^2 + \frac{1}{2}\frac{\dot{S}}{S}\frac{\dot{T}}{T}\right] \tag{12}$$

$$\frac{3H''}{HF}-\frac{3}{2}\frac{H'}{H}\frac{F'}{F^2}=T\left[\frac{\ddot{T}}{2T}+\frac{3}{2}\frac{\dot{S}}{S}\frac{\dot{T}}{T}-\frac{1}{4}\left(\frac{\dot{T}}{T}\right)^2\right] \tag{13}$$

$$\frac{\dot{S}}{S}\frac{H'}{H}=\frac{1}{2}\frac{\dot{T}}{T}\frac{H'}{H} \tag{14}$$

Let us look at (14) first. If $H'=0$, (14) is identically satisfied and the other equations also do not depend on F'. All ψ dependence goes away, giving a metric with a $\partial/\partial\psi$ Killing vector, as discussed by Chodos and Detweiler (1980). We willl return to this case later. We are primarily interested in having a metric that depends on ψ, and therefore let us assume $H'\neq 0$. Then (14) rather remarkably gives

$$\frac{\dot{S}}{S}=\frac{1}{2}\frac{\dot{T}}{T} \tag{15}$$

or

$$S(t)=AT(t)^{1/2} \tag{16}$$

where A is a constant. Thus, if $S(t)$ is expanding in a Hubble expansion, so must the fifth dimension be expanding, rather than shrinking as in the Chodos and Detweiler (1980) case. Note that (14) arose in the $5t$ sector of (5), so this result also depends on $S_{5t}=0$, where $S_{AB}\equiv T_{AB}-\frac{1}{2}g_{AB}T^c_c$, but is otherwise quite general, not depending on the details of the energy-momentum tensor. We of course are letting $T_{AB}=0$. Also note that this does not mean that the distance around the fifth dimension D_5 must be large, since this also depends on $F(\psi)$ in (10).

The final result for D_5 does not depend on the time-dependent equations. However, for completeness, let us solve these equations to make sure they are self-consistent. Using (15), we find the solution of (11) to be

$$S(t)=(C+Dt^2)^{1/2} \tag{17}$$

where C and D are constant. $T(t)$ is given by (16). The Hubble constant is

$$H_0\equiv\left.\frac{\dot{S}}{S}\right|_{t=t_0}=\frac{Dt_0}{C+Dt_0^2} \tag{18}$$

and the deceleration parameter is

$$q_0=-\left.\frac{\ddot{S}S}{\dot{S}^2}\right|_{t=t_0}=-\frac{C}{Dt_0^2} \tag{19}$$

where t_0 is the time of the present epoch.

Let us now turn to the ψ-dependent equations (12) and (13). We see that (13) can be separated and written as

$$\frac{3H''}{HF}-\frac{3}{2}\frac{H'}{H}\frac{F'}{F^2}=C_0 \tag{20}$$

and

$$\frac{\ddot{T}}{2T}+\frac{1}{2}\left(\frac{\dot{T}}{T}\right)^2=\frac{C_0}{T} \tag{21}$$

where C_0 is a constant. Also rather miraculously (12) separates, but only because (16) holds. Then (12) can be written as the two equations

$$\frac{-2k}{H^2A^2}+\frac{2}{F}\left(\frac{H'}{H}\right)^2=C_1 \tag{22}$$

and

$$C_1+\frac{C_0}{3}=\left[\frac{\ddot{S}}{S}+3\left(\frac{\dot{S}}{S}\right)^2\right]T \tag{23}$$

where C_1 is a constant and use has been made of (20) and (15). Now using (15) and (16), we see that (23) is the same as (21) if $C_0=\frac{3}{2}C_1$. Also, the solution for $S(t)$ in (17) also satisfies (21) and (23) if $C=0$ and $D=C_0A^2/3$. Thus the time-dependent equations are all self-consistent and we have

$$S(t)=\left(\frac{C_0}{3}\right)^{1/2}At \tag{24}$$

and $T(t)$ given by (16). Also, (18) gives $H_0=1/t_0$ and (19) gives $q_0=0$.

The ψ dependence in the problem is now determined by (20) and (22), where $C_0=\frac{3}{2}C_1$. We can now solve (22) for $F(\psi)$ and find

$$F=3\left(\frac{H'}{H}\right)^2\left(C_0+\frac{3k}{H^2A^2}\right)^{-1} \tag{25}$$

where we assume that the denominator does not vanish. If this denominator vanishes, we are forced back to the uninteresting $H'=0$ case by (22). If we differentiate (25) with respect to ψ and put the resulting F and F' into (20), we find that (20) is identically satisfied. Thus, $H(\psi)$ is unspecified by our equations. A choice of $H(\psi)$ in the original metric (2) with (10) inserted corresponds to a choice of ψ coordinate system. We will calculate the distance around the fifth dimension in the next section in order to look at $H(\psi)$ effects. Note that we have explicitly assumed that $H'\neq 0$ in order to derive (25). We are interested in $H'\neq 0$ in order to study effects of ψ dependence and the lack of a $\partial/\partial\psi$ killing vector in the metric.

4. CALCULATION OF THE DISTANCE AROUND THE FIFTH DIMENSION

We are now in a position to calculate the distance around the fifth dimension, D_5, which is also the quantity needed if we are to calculate the fine structure constant α. From the form of the metric in (2) and using (10) and (25), we have that

$$D_5 = T(t)^{1/2} \int_0^{\chi} H'\left(\frac{3}{C_0 H^2 + 3k/A^2}\right)^{1/2} d\psi \tag{26}$$

where C_0 and A are nonzero integration constants defined earlier and it is assumed that the fifth dimension is compact with the coordinate ψ ranging from 0 to χ. We now use the earlier assumption that the topology of the fifth dimension is the same as that of a circle. Then the points $\psi = 0$ and $\psi = \chi$ are identified and we must have $H(\psi) = H(\psi + \chi)$. We can integrate (26) to give

$$D_5 = 0 \tag{27}$$

since the integrand is a perfect differential and since $H(0) = H(\chi)$. This result holds for all $k = 0, \pm 1$. Thus we can conclude rather generally in this classical calculation that the distance around the fifth dimension is zero, in the general case where a $\partial/\partial\psi$ killing vector is not assumed to exist for the five-dimensional metric. Notice that the time dependence $T(t)$ plays no role in this conclusion. Because of the unexpected nature of this result, I list the key assumptions that went into it:

1. We are interested in looking at the general case where a $\partial/\partial\psi$ Killing vector does not exist. If $H'(\psi) = 0$, such a Killing vector does exist. Thus we assumed $H'(\psi) \neq 0$.
2. We assumed an initial form of the metric with a maximally symmetric three-dimensional subspace and with a global time coordinate that could serve as the time coordinate of a Gaussian coordinate system.
3. We considered the case where the five-dimensional energy-momentum tensor $T_{AB} = 0$, for simplicity. This could be relaxed to include a generalization to five dimensions of the usual four-dimensional expression for a comoving perfect fluid. Source terms that are time dependent have no effect on our conclusion so long as $S_{5t} = 0$. Conceptually, explicit source terms in a unified theory seem out of place.
4. We took the fifth dimension to be compact with the topology of a circle. This circle was assumed to be small compared to the usual accessible space-time distances.

Chodos and Detweiler (1980) find very different behavior. They find a fifth dimension that contracts in time and has a finite distance around at any given instant. We find $D_5 = 0$ and a time dependence for the fifth dimension that follows the Hubble expansion. How can these be reconciled? If we assume the existence of a $\partial/\partial\psi$ Killing vector [the Kasner solution used by Chodos and Detweiler (1980) has no ψ dependence], then our $H'(\psi) = 0$. The important relation (15) between $\dot{S}$ and $\dot{T}$ no longer follows from the $5t$ sector field equation (14). Our time-dependent equations are then substantially altered and (25) no longer holds. Since the original form of our metric is compatible with the form used by Chodos and Detweiler (1980) if we let $H(\psi) = F(\psi) = 1$, we can reproduce their results, which have $k = 0$.

To conclude, I have shown that if we do not assume the existence of a $\partial/\partial\psi$ Killing vector, then the distance around the fifth dimension is zero, classically. (See the other assumptions above.) This provides an alternative explanation of why the fifth dimension is not observed: it does not shrink in time as in Chodos and Detweiler (1980), but rather always has physical size zero from the constraints placed upon it by the field equations and from the assumed circular topology. I had hoped to calculate D_5 and hence the fine structure constant α, since the global scale invariance, which is present in the usual theory with the Killing vector assumed, is absent in the present case. (This scale invariance, of course, prevents an actual calculation of D_5 in the usual theory.) The fact that in this much more general case where we do not assume the existence of a $\partial/\partial\psi$ Killing vector we get $D_5 = 0$ classically strongly suggests that a proper calculation of α will involve quantum gravity. This is also indicated by the work of Appelquist and Chodos (1983).

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